MATH 4010 | 05/02/2015

Tutorial 4

Eg 1. $(\ell^p)^* = \ell^q$, for $1 < p < +\infty$, $\frac{1}{p} + \frac{1}{q} = 1$

Remark: Here and henceforth, "=" is in the sense of isomorphism, (i.e. $\exists$ bijective linear operator $T: X \to Y$ s.t. $\|Tx\| = \|x\|, \forall x \in X$.)

Idea of proof:

Pf: (i) $((\ell^p)^* \subset \ell^q)$: $\forall f \in (\ell^p)^*$, $\exists\ y_f = \{\eta_j\} \in \ell^q$ s.t. $\forall\ x = \{\xi_j\} \in \ell^p$,

$$f(x) = \sum_{j=1}^{\infty} \xi_j \eta_j \text{ and } \|y_f\|_{\ell^q} \leq \|f\|$$

(ii) $(\ell^q \subset (\ell^p)^*)$: $\forall\ y = \{\eta_j\} \in \ell^q$, construct a functional $f \in (\ell^p)^*$ s.t.

$$f(x) = \sum_{j=1}^{\infty} \xi_j \eta_j, \forall\ x = \{\xi_j\} \in \ell^p \text{ and } \|f\| \leq \|y\|_{\ell^q}$$

Now, we give the proof.

(i) Since $e_k = \{\delta_{kj}\}$, which is a sequence with k-th term 1, others zero, is a Schauder basis of ℓ^p, there exist $\{\xi_k\}$ s.t.

$$x = \sum_{k=1}^{\infty} \xi_k e_k, \ \forall x \in \ell^p.$$

Then, $\forall f \in (\ell^p)^*$, $f(x) = \sum_{k=1}^{\infty} \xi_k f(e_k)$, since f is linear and continuous.

Set $\eta_k = f(e_k)$, thus $f(x) = \sum_{k=1}^{\infty} \xi_k \eta_k$.

It sufficies to show that $\{\eta_k\} \in \ell^q$ and $\|\{\eta_k\}\|_{\ell^q} \leq \|f\|$.

Indeed, for any $n \in \mathbb{N}$, we can construct a sequence $x^{(n)} \stackrel{\triangle}{=} \{\xi_k^{(n)}\}$ as

$$\xi_k^{(n)} = \begin{cases} |\eta_k|^q / \eta_k & \text{if } k \leq n \text{ and } \eta_k \neq 0 \\ 0 & \text{if } k > n \text{ or } \eta_k = 0. \end{cases}$$

Then, it is clear that $x^{(n)} \in \ell^p$, since it has only finite nonzero terms and $f(x^{(n)}) = \sum_{k=1}^{\infty} \xi_k^{(n)} \eta_k = \sum_{k=1}^{n} |\eta_k|^q$.

By the boundness of f, one has

$$\sum_{k=1}^{n} |\eta_k|^q = |f(x^{(n)})| \leq \|f\| \, \|x^{(n)}\|_{\ell^p} \leq \|f\| \Big(\sum_{k=1}^{\infty} |\xi_k^{(n)}|^p \Big)^{\frac{1}{p}}$$

$$\leq \|f\| \Big(\sum_{k=1}^{n} |\eta_k|^{(q-1)p} \Big)^{\frac{1}{p}} = \|f\| \Big(\sum_{k=1}^{n} |\eta_k|^q \Big)^{\frac{1}{p}}$$

So, $\Big(\sum_{k=1}^{n} |\eta_k|^q \Big)^{\frac{1}{q}} \leq \|f\|$

Let $n \to +\infty$, one has $\Big(\sum_{k=1}^{\infty} |\eta_k|^q \Big)^{\frac{1}{q}} = \|\{\eta_k\}\|_{\ell^q} \leq \|f\|$

(ii) $\forall\, y=\{\eta_k\}\in \ell^q$, define a mapping f as follows
$$f(x)=\sum_{k=1}^{\infty}\xi_k\eta_k,\quad \forall x=\{\xi_k\}\in \ell^p.$$
Then, it is easy to check that f is linear.

Moreover, the Hölder inequality yields that
$$|f(x)|\leqslant\Big(\sum_{k=1}^{\infty}|\xi_k|^p\Big)^{\frac{1}{p}}\Big(\sum_{k=1}^{\infty}|\eta_k|^q\Big)^{\frac{1}{q}}$$
which implies that $\|f\|\leqslant\|y\|_{\ell^q}<+\infty$, so f is bounded.

Therefore, we conclude that $f\mapsto y$ is a bijective linear operator and $\|f\|=\|\{\eta_k\}\|_{\ell^q}$. That is $(\ell^p)^*=\ell^q$, for $1<p<+\infty$, $\frac{1}{p}+\frac{1}{q}=1$.

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Eg 2. $(\ell^1)^*=\ell^\infty$

Pf: (i) $\forall f\in(\ell^1)^*$, since $x=\sum_{k=1}^{\infty}\xi_k e_k$ for some ξ_k, where $\{e_k\}$ is a basis of ℓ^1, $f(x)=\sum_{k=1}^{\infty}\xi_k f(e_k)$. Set $\eta_k=f(e_k)$.

It sufficies to prove that $\{\eta_k\}\in\ell^\infty$. Indeed
$$|\eta_k|=|f(e_k)|\leqslant\|f\|\,\|e_k\|_{\ell^1}=\|f\|,\ \forall k\in\mathbb{N}.$$
That is $\|\{\eta_k\}\|_{\ell^\infty}=\sup_k|\eta_k|\leqslant\|f\|<+\infty$.

(ii) $\forall\, y=\{\eta_k\}\in\ell^\infty$, define $f:\ell^1\to\mathbb{R}$ as
$$f(x)=\sum_{k=1}^{\infty}\xi_k\eta_k,\ \forall\, x=\{\xi_k\}\in\ell^\infty.$$
Then f is linear and $|f(x)|\leqslant\sup_k|\eta_k|\sum_{k=1}^{\infty}|\xi_k|$, i.e. $\|f\|\leqslant\|y\|_{\ell^\infty}<+\infty$.

Therefore $\|f\|=\|y\|_{\ell^\infty}$.

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Qn: $(\ell^\infty)^* \overset{?}{=} \ell^1$

Hint: $(\ell^\infty)^*\neq\ell^1$!

Note that ℓ^1 is seperable, but ℓ^∞ is not seperable!

If X^* is seperable, then X is seperable.

So, we have a contradiction!

Please construt a counter-example to show this!

Eg 3. $(c_0)^* = \ell'$, where c_0 is the space of all sequences of scalars converging to zero, with norm $\|\{\xi_k\}\|_{c_0} = \sup_k \|\xi_k\|$.

Pf: (i) $\forall f \in (c_0)^*$, $f(x) = \sum_{k=1}^{\infty} \xi_k f(e_k)$ with $x = \sum_{k=1}^{\infty} \xi_k e_k \in c_0$ with $\xi_k \xrightarrow[k\to+\infty]{} 0$.

Set $\eta_k = f(e_k)$, then $f(x) = \sum_{k=1}^{\infty} \xi_k \eta_k$.

It suffices to show that $\{\eta_k\} \in \ell'$ and $\|\{\eta_k\}\|_{c_0} \leq \|f\|$

Indeed, we can construct $x^{(n)} = \{\xi_k^{(n)}\}$ as

$$\xi_k^{(n)} = \begin{cases} \frac{|\eta_k|}{\eta_k} & \text{if } k \leq n \text{ and } \eta_k \neq 0 \\ 0 & \text{if } k > n \text{ or } \eta_k = 0. \end{cases}$$

Then $f(x^{n}) = \sum_{k=1}^{\infty} \xi_k^{(n)} \eta_k = \sum_{k=1}^{n} |\eta_k|$, $x^{(n)} \in c_0$

and $|f(x^{n})| \leq \|f\| \|x^{(n)}\|_{c_0} \leq \|f\|$

So $\sum_{k=1}^{n} |\eta_k| \leq \|f\| < +\infty$, i.e. $\{\eta_k\} \in \ell'$

(ii) $\forall y = \{\eta_k\} \in \ell'$, define $f: c_0 \to \mathbb{R}$ as

$$f(x) = \sum_{k=1}^{\infty} \xi_k \eta_k, \quad \forall x = \{\xi_k\} \in c_0$$

Then $|f(x)| \leq \sup_k |\xi_k| \sum_{k=1}^{\infty} |\eta_k| \leq \|x\|_{c_0} \|\{\eta_k\}\|_{\ell'}$

So $\|f\| \leq \|\{\eta_k\}\|_{\ell'} < +\infty$.

Therefore $\|f\| = \|\{\eta_k\}\|_{\ell'}$ with $\eta_k = f(e_k)$.